Hidden Variable Theory

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Abstract

These are my personal notes on hidden variable theory, a keen interest of mine, based on lectures and conversations by Richard Scalettar, discussions with my advisor, Steve Carlip and various papers written by John S. Bell.

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1 Introduction To Hidden Variable Theory

Most physicists say that measurements do *not* reveal a pre-existing value of the measured quantity; we say that the outcome of the measurement is brought into being by the very act of measurement. There are two reasons why we say this:

1. **Measurements disturb the system**

   If we want to measure the position of an atom we must shine light on it. The wavelength of the light must be about the atom's size: $\lambda \lesssim \Delta x$. But the light imparts a momentum to the atom: $p_x = h/\lambda \gtrsim h/\Delta x$, thereby changing the very thing we want to measure.

   But this is a false argument because it's non-sequitar. It begs the question: So what? Logically, the fact that we can't measure x and p_x simultaneously doesn't mean that x and p_x didn't exist simultaneously to begin with!

   Perhaps x and p_x really do simultaneously exist, but we just can't measure them simultaneously. Therefore, while it's true that the very act of measurement disturbs the thing we want to measure, but that doesn't mean that pre-determined values didn't exist to begin with.

2. **Uncertainty Principle Forbids It**

   The uncertainty principle states that it's impossible to prepare an ensemble[1] of systems with an arbitrarily sharp distinction of x and p_x.

   However, the uncertainty principle does *not* say that an individual member of that ensemble cannot have a definite value of x and p_x!

Therefore, there's nothing about the formalism of quantum mechanics that says a-priori that a definite value does not exist before the measurement.

One might be tempted to ask, "who cares?" If you can't measure x and p_x simultaneously, even if both values exist simultaneously, the whole debate is moot! To address this attitude, it might be useful to consider an analogy.

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[1]How did the word "ensemble" creep into the uncertainty principle? The uncertainty principle says $\Delta x \Delta p_x \gtrsim \hbar/2$, where $(\Delta x)^2 = \langle x^2 \rangle^2 - \langle x \rangle^2$. The $\langle \rangle$ operator is actually defined by preparing an ensemble of systems.

Consider statistical mechanics. It's impossible to know or even store on all the computers on Earth the positions and momenta for all 10^{23} coffee molecules in your coffee mug. Does that mean statistical mechanics is useless? Certainly not! In fact, it's one of the most useful branches of physics we have! Just as constraints from classical mechanics makes statistical mechanics useful, constraints from a hidden variable theory might prove useful in quantum mechanics. If there were a HVT, and even if we still couldn't measure x and p_x simultaneously, knowing the theory would vastly improve our understanding of quantum mechanics.

In a HVT, the state function $|\Psi\rangle$ is composed of an ensemble of systems, each with definite values for all observables which exist before a measurement is made. You still need probability because before the measurement is made, you don't know which one of the systems in the ensemble you'll get. The uncertainty principle still exists in a HVT. However, uncertainty principle becomes a statement about what types of ensembles it's possible to prepare. Like statistical mechanics. And yet unlike statitstical mechanics.

How do we go about proving that a hidden variable theory exists? We construct one! In fact, we do just that in section 2. How do we prove that no HVT exists? We create an impossibility proof. In fact, we do just that in section 4.

1.1 Interpretations of Quantum Mechanics

Quantum mechanics throws indeterminancy into act of measurement.[2] This is often disturbing to people who learn from an early age that human beings were meant to classify and quantify everything in the universe. Even if we were to know everything there is to know about a particle, its state function, we cannot predict with certainty the outcome of a measurement on the particle's position. The only thing quantum mechanics has to offer is statistical information about the possible outcomes of a position measurement. The efforts of the best minds humanity has to offer has been turned to the question of whether this peculiarity is due to some fault of the measuring apparatus

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[2]A huge misconception is that quantum mechanics is stochastic by nature. This is false. Schrödinger's equation is completely deterministic. It's only the act of measurement that throws probability into the theory. But measurement has more to do with humans fiddling with a system than it does the postulates of quantum mechanics.

or a fundamental indeterminancy in nature.

Suppose we make a position measurement of an electron and found it at point P. An interesting question to ask is: where was the particle *just* before the measurement was made? There are many positions on this question:

Realism

The particle was at point P. This is the most reasonable position and the one that Einstein advocated. Quantum mechanic's indeterminacy is not a fact of nature but simply a statement of our ignorance or inability to manufacture a measuring device which doesn't disturb the system.

Since nobody in their right mind would deny the success of QM, realists maintain that one of two things must be true:

1. QM is an incomplete theory.
2. QM is an excellent model, but with no semblance to reality.

Many realists believe that QM is an incomplete theory which needs only a minor adjustment: a hidden variable. It's called a hidden variable because so far, we haven't been able to detect or theorize what this variable could be.

Orthodoxy

The particle wasn't really anywhere; we forced the position to come into existance by the very act of measurement. Measurements not only disturb what they measure. . .they produce it! We compel the particle to assume a definite position. This is called the *Copenhagen Interpretation* of quantum mechanics.

If this position is correct, then the act of measuring something is a very strange process. We get into problems of who is allowed to make a measurement and what a measurement consists of.

Some people say that the notion of HVT is fringe physics. To these people, I say that the Copenhagen interpretation is nothing more than metaphysics and magic.

Agnosticism

> Agnostics refuse to answer the question. Pauli advocated this point of view by saying "One should no more rack one's brain about the problem of whether something one cannot know anything about exists...then about the ancient question of how many angels are able to sit on the point of a needle[3]'.

John S. Bell formulated Bell's Theorem which eliminated agnosticism from the viable choice of interpretations and made the choice of realism vs orthodoxy a matter of experiment. Alan Aspect[4] and others performed the experiment, and so far, all the evidence seems to point towards orthodoxy.

By the way, what if we were to make a second position measurement of the electron at point P immediately after the first position measurement? Everyone agrees (including experiment) the particle will still be at point P. How does indeterminacy account for this? This particular issue is so important that there's a postulate devoted to it: the "wavepacket reduction postulate" (WRP), or the "R process". As soon as the position measurement is made, the wavefunction collapses to a single eigenstate so that there's no longer indeterminacy in the particle's location. However, soon after the first measurement is made, the wavefunction begins to spread out according to Schrödinger's equation, so the second measurement must be made very quickly if you want to find the electron still at P with absolute certainty.

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[3]Quoted by N. David Mermin, *Is the moon there when nobody looks?*, Physics Today, April 1985, p. 40

[4]A. Aspect, P. Grangier, G. Roger, *Phys. Rev. Lett.* **49**, 91 (1982)

2 A Successful Hidden Variable Theory

2.1 Before we begin

We'll construct a successful hidden variable theory in two dimensions, which was first proposed by John S. Bell himself. To begin, we'll review some properties of 2 dimensional Hilbert spaces. Proofs are given in appendix B.

1. Any hermitian operator $\hat{A}$ can be written as a linear combination of the identity operator plus the three Pauli spin matrices:

$$\hat{A} \;=\; a_0\hat{I} + a_i\sigma^i \;=\; a_\mu\sigma^\mu \qquad \{a_\mu\} \in Z \tag{1}$$

2. Any observable of the form $\hat{A} = a_\mu\sigma^\mu$ has eigenvalues of:

$$\nu(\hat{A}) = a_0 \pm |\vec{\mathbf{a}}| \qquad \text{where } |\vec{a}| = \sqrt{a_\mu a^\mu} \tag{2}$$

3. The operators $\hat{A}$ and $\hat{B}$ commute if $\vec{\mathbf{a}}$ is parallel to $\vec{\mathbf{b}}$.

4. Every state $|\Psi\rangle = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$ is an eigenstate of $\sigma_n = \vec{\sigma}\cdot\hat{n}$ for some $\hat{n}$.

2.2 Construction of the theory

Let $|\Psi^n\rangle$ be a state given by an ensemble of systems and characterized by $\hat{n}$ from property 4. Let each system in the ensemble have a hidden variable associated with it called $\hat{m}$, with any direction of $\hat{m}$ being equally likely.

Consider the operator $\hat{A}$. A measurement of $\hat{A}$ in any given system of the ensemble is determined by the value $\hat{m}$ for that system, and in 2D, there are only two possible outcomes of the measurement:

$$\nu(\hat{A}) = \begin{cases} a_0 + |\vec{\mathbf{a}}| & \text{if } (\hat{m} + \hat{n})\cdot\vec{\mathbf{a}} > 0 \\ a_0 - |\vec{\mathbf{a}}| & \text{if } (\hat{m} + \hat{n})\cdot\vec{\mathbf{a}} < 0 \end{cases}$$

The result of a measurement of $\vec{\mathbf{A}}$ is pre-determined by the existing value of $\hat{m}$ in that particular system in the ensemble. What we need to do is to show that $\langle\hat{A}\rangle$ in our toy HVT has the same value predicted by quantum

mechanics without a hidden variable. If we show this, then our toy hidden variable theory is a viable theory.

So we're now how to calculate $\langle \vec{A} \rangle$. Given that $\vec{A} = a_0 \hat{I} + \vec{a} \cdot \sigma$ and property 4:

$$\hat{n} \cdot \vec{\sigma} |\Psi\rangle = \pm |\Psi\rangle \tag{3}$$

$$\langle \Psi | \vec{A} | \Psi \rangle = a_0 + \vec{a} \cdot \hat{n} \tag{4}$$

Choose $\hat{z}$ to be parallel with $\vec{a}$. Then:

$$\vec{a} \cdot (\hat{m} + \hat{n}) = (m_z + n_z)|\vec{a}|$$

Of course the $|\vec{a}|$ has no bearing on whether $\vec{a} \cdot (\hat{m} + \hat{n})$ and hence the eigenvalue we obtain from measuring $\vec{A}$. Now, $\hat{m}$ is equally likely to point in any direction, so:

$$\hat{m} = \sin(\theta)\cos(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\theta)\hat{z}$$

So let's find the average eigenvalue. We're averaging over a sphere, so we need to divide by 4π.

$$\langle \vec{A} \rangle = \frac{1}{4\pi} \int_0^\pi \sin(\theta)\, d\theta \int_0^{2\pi} d\phi \begin{cases} a_0 + |\vec{a}| & \cos(\theta) > -n_z \\ a_0 - |\vec{a}| & \cos(\theta) < -n_z \end{cases}$$

$$= \frac{1}{2} \int_0^\pi \sin(\theta) \begin{cases} a_0 + |\vec{a}| & \cos(\theta) > -n_z \\ a_0 - |\vec{a}| & \cos(\theta) < -n_z \end{cases}$$

Let let θ_0 be the angle such that $-n_z = \cos(\theta_0)$.

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$$\langle \vec{\mathbf{A}} \rangle = a_0 + \frac{|\vec{\mathbf{a}}|}{2} \left[\int_0^{\theta_0} \sin(\theta)\, d\theta - \int_{\theta_0}^{\pi} \sin(\theta)\, d\theta \right]$$

$$= a_0 + \frac{|\vec{\mathbf{a}}|}{2} \left[-\cos(\theta)|_0^{\theta_0} + \cos(\theta)|_{\theta_0}^{\pi} \right]$$

$$= a_0 - |\vec{\mathbf{a}}| \cos(\theta_0)$$

$$= a_0 + |\vec{\mathbf{a}}| n_z$$

$$= a_0 + \vec{\mathbf{a}} \cdot \hat{n}$$

Just as expected. We showed that the expectation value of $\vec{\mathbf{A}}$ is what we'd expect it to be, in full agreement with what standard quantum mechanics predicts. Thus, as strange as it seems, we have constructed a correct hidden variable theory.

3 Von Neumann's Silly Proof

Now that I've proved a successful HVT exists, I'm going to prove that no successful HVT exists. But first, I want to show you a false proof that no HVT exists formulated by John VonNeumann in 1932. A physicist named Grete Hermann pointed out the logical flaw in his proof, but nobody listened to her. It wasn't until John S. Bell, who had already made a name for himself, had pointed out the flaw that people began to realize Von Neumann's error. Before starting, we need to know some of the plausible features that a HVT might have.

3.1 What Are Plausible Features That A HVT Might Have?

Imagine we're doing quantum mechanics and we have the operators $\hat{A}$, $\hat{B}$, $\hat{C}$ and a state $|\phi\rangle$ which is an *ensemble* of systems, each with definite preexisting values of the measurements $\nu(\hat{A})$, $\nu(\hat{B})$ and $\nu(\hat{C})$. Suppose we have a valid relationship among these operators:

$$f(\hat{A}, \hat{B}, \hat{C}) = 0 \tag{5}$$

If the operators $\hat{A}$, $\hat{B}$, $\hat{C}$ all commute then we would expect the same relationship among the simultaneous eigenvalues:

$$f(\nu(\hat{A}), \nu(\hat{B}), \nu(\hat{C})) = 0 \tag{6}$$

This is a variable constraint to put on the HVT, and it seems quite reasonable. This constraint is often used by HVT people, but Von Neumann tried using it to disprove HVT.

3.2 The Proof

In 1932, Von Neumann tried giving a silly proof that no HVT existed by assuming that such theories must obey equations 5 and 6 even if the operators don't commute. This is silly because we can't measure the eigenvalues simul-

taneously so we don't really expect this constraint to be true if the operators don't commute.

Let $\hat{A} = \sigma_x$, $\hat{B} = \sigma_y$ and $\hat{C} = \sigma_x + \sigma_y$. Then certainly, $\hat{C} - \hat{B} - \hat{A} = 0$. But we know that $\nu(\hat{A}) = \nu(\hat{B}) = \pm 1$, so $\nu(\hat{A}) + \nu(\hat{B}) = -2, 0, 2$.

However[5], $\nu(\hat{C}) = \nu(\hat{A} + \hat{B}) = \pm\sqrt{2}$ which is certainly not equal to $-2, 0, 2$.

So Von Neumann had an operator relationship of the form $\hat{C} - \hat{B} - \hat{A} = 0$, but $\nu(\hat{C}) - \nu(\hat{A}) - \nu(\hat{B}) \neq 0$, so no HVT. But the proof is silly since you shouldn't expect this to be true for operators that don't commute. The eigenvalues don't exist simultaneously.

People quoted Von Neumann's proof for 34 years until 1966 when John S. Bell pointed out that the proof is silly. Here is what Bell had to say about Von Neumann's proof:

> Yet the Von Neumann proof, if you actually come to grips with it, falls apart in your hands. There's nothing to it. It's not just flawed...it's silly. When you translate his assumptions into physical significance, they're nonsense. You may quote me on this. The proof of Von Neumann is not just false, it's foolish...

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[5]See Appendix A for a proof.

4 Proof that no HVT Exists

4.1 Introduction

As mentioned, if it were impossible to simultaneously measure eigenvalues for operators that don't commute, even if they exist simultaneously, knowledge of a HVT would bring us to a closer understanding of nature, which is one of the aims of physics. Bell's theorem is the standard proof that no HVT exists. The theorem shows that one of four possibilities must be true:

1. Nonlocality of physics: faster than light communication is possible. This is what physicists call 'spooky action at a distance'.

2. A many world interpretation of quantum mechanics: all outcomes that can occur for a particular measurement do occur in different realities.

3. Strong determinism: This basically says that that quantum mechanics, while a useful model, is just plain wrong.

We'll begin by

A Properties Of A 2 Dimensional Hilbert Space

In this appendix, I prove four theorems used to construct our successful HVT in two dimensions. Let the generalized four Pauli spin matrices be $\vec{\sigma}^{(4)} = (I, \sigma_x, \sigma_y, \sigma_z)$.

A.1 Proof of Property 1

We want to prove that any 2 dimensional hermitian operator can be written as a linear combination of the 4 Pauli spin matrices.

Proof by construction: if we can write down the most general hermitian operator as a combination of the Pauli spin matrices, we've proved the assertion.

The most general hermitian operator can be written as:

$$\hat{A} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

which, after a little algebra, can be easily verified to equal

$$\mathbf{\vec{A}} = \frac{A + D}{2}\hat{I} + \frac{A - D}{2}\sigma_x + \frac{A - D}{2}\sigma_z + B\sigma_x + C\sigma_y$$

Therefore, the assertion is proved.

A.2 Proof of property 2

We want to prove that the eigenvalues of Pauli spin matrices in an arbitrary direction is still $\lambda = \pm 1$.

If $\hat{n}$ is a unit vector, the Pauli spin matrices along $\hat{n}$ is given by:

$$\vec{\sigma} \cdot \hat{n} = \vec{\sigma} \cdot (n_x \hat{x} + n_z \hat{z} + n_z \hat{z})$$

$$= n_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + n_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + n_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$$

Since $|\hat{n}| = 1$, the determinant of $\vec{\sigma} \cdot \hat{n}$ must be the same as the determinant of $\vec{\sigma}$, as illustrated here:

$$\det(\vec{\sigma} \cdot \lambda) = \begin{vmatrix} n_z - \lambda & n_x - in_y \\ n_x + in_y & -n_z - \lambda \end{vmatrix}$$

$$= \lambda^2 - (n_x^2 + n_y^2 + n_z^2)$$

$$= \lambda^2 - 1 \implies \lambda = \pm 1$$

As expected, $\det(\sigma \cdot \hat{n}) = \pm 1$. Therefore, the determinant of $\sigma_x + \sigma_y$ is:

$$\det(\sigma_x + \sigma_y) = \det(\vec{\sigma} \cdot \hat{n})$$

where $\hat{n} = (1,1,0)\sqrt{2}$, which by the above argument must be $\lambda = \pm 1$.

Therefore, the determinant of $a^\mu \sigma_\mu$ is:

$$\det(a^\mu \sigma_\mu) = \det(a_0 \hat{I} + a_i \sigma^i) = \det(a_0 \hat{I}) + \det(a_i \sigma^i) = a_0 \pm |\vec{a}|$$

which proves the assertion.

A.3 Proof of Property 3

We'd like to prove that the commutator of operators $\hat{A}$ and $\hat{B}$ is zero if $\vec{\mathbf{a}}$ is parallel to $\vec{\mathbf{b}}$.

Let $\hat{A}$ and $\hat{B}$ be two dimensional hermitian operators. From property 2 we can write $\hat{A} = a_0 \hat{I} + \vec{\sigma} \cdot \vec{\mathbf{a}}$ and $\hat{B} = b_0 \hat{I} + \vec{\sigma} \cdot \vec{\mathbf{b}}$. Since the identity operator commutes with everything, we can write:

$$[\vec{\mathbf{A}}, \vec{\mathbf{B}}] = [a_0 \hat{I} + \vec{\mathbf{a}} \cdot \vec{\sigma}, \ b_0 \hat{I} + \vec{\sigma} \cdot \vec{\mathbf{b}}] = [\vec{\mathbf{a}} \cdot \vec{\sigma}, \ \vec{\sigma} \cdot \vec{\mathbf{b}}] = [\sigma_i a^i, \ \sigma_i b^i]$$

But if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are parallel, then $a^i = cb^i$ for some number c:

$$[\vec{\mathbf{A}}, \vec{\mathbf{B}}] = c[\sigma_i a^i, \ \sigma_i a^i] = c \, a^i [\sigma_i, \ \sigma_i]$$

which must be 0 since each spin matrix commutes with itself. Therefore, the assertion is proved.